

LINEAR RELATIONS BETWEEN $\mathcal{N} \geq 4$ SUPERGRAVITY AND SUBLEADING-COLOR SYM AMPLITUDES

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ABSTRACT

The IR divergences of supergravity amplitudes are less severe than those of planar SYM amplitudes, and are comparable to those subleading-color SYM amplitudes that are most subleading in the $1/N$ expansion, namely $\mathcal{O}(1/\epsilon^L)$ for L -loop amplitudes. We derive linear relations between one- and two-loop four-point amplitudes and one-loop five-point amplitudes of $\mathcal{N} = 4, 5$, and 6 supergravity and the most-subleading-color contributions of the analogous amplitudes of $\mathcal{N} = 0, 1$, and 2 SYM theory, extending earlier results for $\mathcal{N} = 8$ supergravity amplitudes. Our work relies on linear relations between $\mathcal{N} \geq 4$ supergravity and planar SYM amplitudes that were recently derived using the double-copy property of gravity, and color-kinematic duality of gauge theories.

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1 Introduction

Recent progress in the understanding of perturbative gauge theory and gravity amplitudes has been fueled by the discovery of a color-kinematic duality of gauge theory amplitudes by Bern, Carrasco, and Johansson [1, 2]. Any L -loop n -point gauge theory scattering amplitude where all particles are in the adjoint representation can be written as

$$\mathcal{A}_n^{(L)} = i^L g^{n-2+2L} \sum_i \int \prod_{l=1}^L \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_i} \frac{n_i c_i}{\prod_{\alpha_i} p_{\alpha_i}^2}. \quad (1.1)$$

The sum runs over a set of L -loop n -point diagrams containing only cubic vertices, the integral is over a set of L loop momenta, the product in the denominator contains all the propagators of the diagram, the n_i are kinematic numerators depending on momenta, polarizations, etc., and the S_i are symmetry factors. The c_i are color factors associated with the diagrams; they are not all linearly independent, but satisfy various (Jacobi) relations. Because of this, the representation (1.1) of an amplitude is not unique; different choices of kinematic numerators yield the same amplitude. This freedom in the choice of n_i is described as a generalized gauge transformation [3].

A given representation (1.1) is said to satisfy color-kinematic (or BCJ) duality if the kinematic numerators n_i obey precisely the same set of algebraic relations observed by the color factors c_i . It was conjectured that all gauge theories possess such a representation [1, 2], and this was verified through three loops for the $\mathcal{N} = 4$ SYM four-point [2, 4] and five-point [5] amplitudes. It was further conjectured [2, 3] that L -loop n -point gravity amplitudes can be constructed from a pair of gauge theory amplitudes as

$$\mathcal{M}_n^{(L)} = i^{L+1} \left(\frac{\kappa}{2}\right)^{n-2+2L} \sum_i \int \prod_{l=1}^L \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_i} \frac{n_i \tilde{n}_i}{\prod_{\alpha_i} p_{\alpha_i}^2}, \quad (1.2)$$

where n_i and \tilde{n}_i are the kinematic numerators of the gauge theory representation (1.1), provided that at least one of the two representations satisfies color-kinematic duality. The $\mathcal{N} = 8$ supergravity four-point [2, 4] and five-point [5] amplitudes were obtained as double copies (1.2) of the corresponding $\mathcal{N} = 4$ SYM amplitudes.

The double-copy property of gravity theories was further used [6, 7] to obtain new expressions for $\mathcal{N} \geq 4$ supergravity amplitudes by combining loop amplitudes from an $\mathcal{N} \leq 4$ SYM theory (which need not satisfy BCJ duality) with the kinematic numerators from an $\mathcal{N} = 4$ SYM theory (whose amplitudes do). Specifically, it was shown in refs. [6, 7] that the supergravity amplitudes on the left side of the following equation

$$\begin{aligned} \mathcal{N} = 8 \text{ supergravity} & : (\mathcal{N} = 4 \text{ SYM}) \times (\mathcal{N} = 4 \text{ SYM}) \\ \mathcal{N} = 6 \text{ supergravity} & : (\mathcal{N} = 4 \text{ SYM}) \times (\mathcal{N} = 2 \text{ SYM}) \\ \mathcal{N} = 5 \text{ supergravity} & : (\mathcal{N} = 4 \text{ SYM}) \times (\mathcal{N} = 1 \text{ SYM}) \\ \mathcal{N} = 4 \text{ supergravity} & : (\mathcal{N} = 4 \text{ SYM}) \times (\mathcal{N} = 0 \text{ SYM}) \end{aligned} \quad (1.3)$$

(where $\mathcal{N} = 0$ SYM denotes pure YM theory) can be represented as double copies of the gauge theories on the right side, which have the same state multiplicities as the corresponding gravity theory.

Gravity amplitudes have IR divergences at loop level [8]; in dimensional regularization, the leading IR divergence of an L -loop amplitude goes as $\mathcal{O}(1/\epsilon^L)$, where $D = 4 - 2\epsilon$ [9]. Moreover, the IR-divergent terms of an L -loop gravity amplitude are given by the exponentiation of the one-loop divergence [8, 10–12]. This behavior has been verified at two loops for $\mathcal{N} = 8$ supergravity [13, 14] and also for $\mathcal{N} \geq 4$ supergravity [7].

The expressions for various supergravity amplitudes obtained in refs. [5–7, 15] involve combinations of planar SYM amplitudes, whose leading IR divergence goes as $\mathcal{O}(1/\epsilon^{2L})$ at L loops. Therefore nontrivial cancellations among the SYM amplitudes are required to match the $\mathcal{O}(1/\epsilon^L)$ leading IR behavior of the supergravity amplitude.

IR divergences of gauge theory amplitudes that are subleading in the $1/N$ expansion are less severe than those of planar amplitudes. In ref. [16, 17] it was shown that the leading IR divergence of $A^{(L,k)}$, which denotes the subleading-color L -loop amplitude suppressed by N^k relative to the planar amplitude, is of $\mathcal{O}(1/\epsilon^{2L-k})$ for $\mathcal{N} = 4$ SYM four-point amplitudes. We will verify in sec. 4 that the same result also holds for subleading-color $\mathcal{N} < 4$ SYM amplitudes.

The leading IR divergence of $A^{(L,L)}$, the most-subleading-color L -loop amplitude, therefore, is $\mathcal{O}(1/\epsilon^L)$, precisely the same as the leading divergence of the L -loop gravity amplitude. One therefore speculates that such amplitudes could provide a basis for expanding gravity amplitudes. In previous work, we showed that the one- and two-loop four-point functions [16] and the one-loop five-point function [18] of $\mathcal{N} = 8$ supergravity are linearly related to the most-subleading-color amplitudes of $\mathcal{N} = 4$ SYM theory.

In this paper, we show that analogous linear relations hold between certain amplitudes of $\mathcal{N} + 4$ supergravity on the one hand, and the most-subleading-color amplitudes of \mathcal{N} SYM theory, with $0 \leq \mathcal{N} < 4$, where the SYM theory is one of the two gauge theories in the double copy representation (1.3). We establish these relations by demonstrating their equivalence to the expressions for $\mathcal{N} \geq 4$ supergravity amplitudes recently obtained in refs. [6, 7].

We point out that the linear supergravity/SYM relations between *integrated* amplitudes found in this paper and in refs. [5–7, 15, 16, 18] involve amplitudes in which the kinematic numerators are independent of loop momenta, and so can be pulled outside the loop integrals. It remains an open question whether similar relations between integrated SYM and supergravity amplitudes can be obtained when the kinematic numerators depend on the loop momenta.

This paper is structured as follows. In section 2, we review the linear relations between $\mathcal{N} + 4$ supergravity amplitudes and planar \mathcal{N} SYM amplitudes obtained in refs. [6, 7], partly to establish notation. In section 3, we derive new linear relations between $\mathcal{N} + 4$ supergravity amplitudes and subleading-color \mathcal{N} SYM amplitudes, generalizing results previously obtained in refs. [16, 18]. In section 4, the leading IR divergences of subleading-color amplitudes for generic $SU(N)$ gauge theories are discussed, and sec. 5 contains some concluding remarks.

2 Review of relations between supergravity and planar SYM amplitudes

The color structure of a gauge theory amplitude may be expressed by decomposing the amplitude in either a trace basis [19] or a basis of color factors [20, 21]. The trace basis is more conducive to exhibiting the $1/N$ expansion of the gauge theory, while the color basis is more natural for exhibiting color-kinematic duality and for writing gravity amplitudes as double copy of gauge theory amplitudes. In this section, we will review the trace basis for loop-level amplitudes, and then summarize the results derived in refs. [6, 7] for one- and two-loop four-point functions and for one-loop five-point functions of $\mathcal{N} \geq 4$ supergravity.

2.1 Trace basis

One-loop n -point amplitudes of particles in the adjoint representation of an $SU(N)$ gauge theory may be decomposed into a basis of single and double traces [19]

$$\begin{aligned} \mathcal{A}^{(1)}(1, 2, \dots, n) &= g^n \sum_{j=1}^{\lfloor n/2 \rfloor + 1} \sum_{\sigma \in S_n / S_{n;j}} A_{n;j}(\sigma(1) \dots \sigma(n)) Gr_{n;j}(\sigma) \\ Gr_{n;1}(\mathbb{1}) &\equiv N \operatorname{Tr}(T^{a_1} \dots T^{a_n}) \\ Gr_{n;j}(\mathbb{1}) &\equiv \operatorname{Tr}(T^{a_1} \dots T^{a_{j-1}}) \operatorname{Tr}(T^{a_j} \dots T^{a_n}) \end{aligned} \quad (2.1)$$

whose coefficients are referred to as color-ordered amplitudes. Here T^a are the generators in the defining representation of $SU(N)$, normalized according to $\operatorname{Tr}(T^a T^b) = \delta^{ab}$, and $S_{n;j}$ is the subgroup of S_n that leaves the trace structure $Gr_{n;j}$ invariant. At higher loops (for $n > 5$), the color decomposition will include triple and higher traces as well, and at L loops, the amplitudes can contain powers of N up to N^L , and so have an expansion in $1/N$.

Since in this paper we focus primarily on four- and five-point functions in supersymmetric Yang-Mills theories (with \mathcal{N} supersymmetries), we specialize the notation for these cases. For five-point functions, we have

$$\begin{aligned} \mathcal{A}_{\mathcal{N}}^{(1)}(1, 2, 3, 4, 5) &= g^5 \left(\sum_{S_5 / \mathbb{Z}_5 \times \mathbb{Z}_2} A_{12345, \mathcal{N}}^{(1,0)} N [\operatorname{Tr}(12345) - \operatorname{Tr}(54321)] \right. \\ &\quad \left. + \sum_{S_5 / \mathbb{Z}_2 \times S_3} A_{12;345, \mathcal{N}}^{(1,1)} \operatorname{Tr}(12) [\operatorname{Tr}(345) - \operatorname{Tr}(543)] \right) \end{aligned} \quad (2.2)$$

where $A^{(1,0)}$ denotes the planar (leading-color) one-loop amplitudes, $A^{(1,1)}$ the subleading-color one-loop amplitudes, and $\operatorname{Tr}(123 \dots) \equiv \operatorname{Tr}(T^{a_1} T^{a_2} T^{a_3} \dots)$. Only single and double traces occur at any number

of loops for four-point functions, so we write

$$\begin{aligned}
\mathcal{A}_{\mathcal{N}}^{(L)}(1, 2, 3, 4) = & g^{2+2L} \Big(A_{1234, \mathcal{N}}^{(L)} [\text{Tr}(1234) + \text{Tr}(1432)] \\
& + A_{1342, \mathcal{N}}^{(L)} [\text{Tr}(1243) + \text{Tr}(1342)] \\
& + A_{1423, \mathcal{N}}^{(L)} [\text{Tr}(1423) + \text{Tr}(1324)] \\
& + A_{13;42, \mathcal{N}}^{(L)} [\text{Tr}(13) \text{Tr}(24)] \\
& + A_{14;23, \mathcal{N}}^{(L)} [\text{Tr}(14) \text{Tr}(23)] \\
& + A_{12;34, \mathcal{N}}^{(L)} [\text{Tr}(12) \text{Tr}(34)] \Big)
\end{aligned} \tag{2.3}$$

The color-ordered amplitudes $A^{(L)}$ may be further decomposed in a $1/N$ expansion of amplitudes $A^{(L,k)}$, with $k = 0, \dots, L$. Specifically, the amplitudes suppressed by even powers of N relative to the leading-color (planar) amplitude contribute to the single-trace amplitudes, and those suppressed by odd powers of N contribute to the double-trace amplitudes

$$A_{ijkl, \mathcal{N}}^{(L)} = \sum_{m=0}^{\lfloor \frac{L}{2} \rfloor} N^{L-2m} A_{ijkl, \mathcal{N}}^{(L, 2m)}, \quad A_{ij;kl, \mathcal{N}}^{(L)} = \sum_{m=0}^{\lfloor \frac{L-1}{2} \rfloor} N^{L-2m-1} A_{ij;kl, \mathcal{N}}^{(L, 2m+1)} \tag{2.4}$$

We will also find it useful to define ratios of loop-level to tree-level amplitudes for four-point functions as follows:

$$M_{SYM, \mathcal{N}}^{(L, 2m)}(s, t) \equiv \frac{(g^2 N)^L A_{1234, \mathcal{N}}^{(L, 2m)}}{A_{1234}^{(0)}}, \quad M_{SYM, \mathcal{N}}^{(L, 2m+1)}(s, t) \equiv -\frac{(g^2 N)^L A_{13;42, \mathcal{N}}^{(L, 2m+1)}}{\sqrt{2} A_{1234}^{(0)}} \tag{2.5}$$

where $s = (k_1 + k_2)^2$, $t = (k_1 + k_4)^2$ and $u = (k_1 + k_3)^2$ are the usual Mandelstam invariants, with k_i the momenta of the external particles, and $A_{1234}^{(0)}$ denotes the tree-level color-ordered amplitude, which is independent of the number \mathcal{N} of supersymmetries.

2.2 Four-point amplitudes

Alternatively, amplitudes may be decomposed in a basis of color factors [20, 21]. One-loop four-point amplitudes of particles in the adjoint representation of a gauge theory may be decomposed in a basis of color factors of the box diagram as

$$\mathcal{A}_{\mathcal{N}}^{(1)}(1, 2, 3, 4) = g^4 \left(c_{1234} A_{1234, \mathcal{N}}^{(1,0)} + c_{1342} A_{1342, \mathcal{N}}^{(1,0)} + c_{1423} A_{1423, \mathcal{N}}^{(1,0)} \right) \tag{2.6}$$

where c_{1234} is obtained by inserting a factor of the $\text{SU}(N)$ structure constants at each vertex of the box diagram

$$c_{1234} = \tilde{f}^{ea_1b} \tilde{f}^{ba_2c} \tilde{f}^{ca_3d} \tilde{f}^{da_4e}. \tag{2.7}$$

By using

$$\tilde{f}^{abc} = i\sqrt{2}f^{abc} = \text{Tr}([T^a, T^b]T^c) \tag{2.8}$$

to express c_{1234} in terms of the trace basis (2.3), one ascertains [21] that the coefficients in the one-loop color basis (2.6) are identical to the planar one-loop color-ordered amplitudes $A_{ijkl,\mathcal{N}}^{(1,0)}$ defined in eq. (2.4).

Specializing to $\mathcal{N} = 4$ SYM, we write the coefficients in eq. (2.6) in the form of eq. (1.1)

$$A_{1234,\mathcal{N}=4}^{(1,0)} = i n_{1234} \mathcal{I}_{1234}^{(1)}, \quad \mathcal{I}_{1234}^{(1)} = \int \frac{d^D p}{(2\pi)^D} \frac{1}{p^2 (p - k_1)^2 (p - k_1 - k_2)^2 (p + k_4)^2} \quad (2.9)$$

where the kinematic numerators n_{1234} (which in this case are independent of the loop momentum and hence come outside the integral) satisfy

$$n_{1234} = n_{1342} = n_{1423} = st A_{1234}^{(0)}. \quad (2.10)$$

Because the numerator factors (2.10) for the $\mathcal{N} = 4$ SYM amplitude satisfy color-kinematic duality, they can be used in a double-copy representation (1.2) of the one-loop four-point amplitude of $\mathcal{N} + 4$ supergravity [6]

$$\begin{aligned} \mathcal{M}_{\mathcal{N}+4}^{(1)}(1, 2, 3, 4) &= i \left(\frac{\kappa}{2} \right)^4 \left(n_{1234} A_{1234,\mathcal{N}}^{(1,0)} + n_{1342} A_{1342,\mathcal{N}}^{(1,0)} + n_{1423} A_{1423,\mathcal{N}}^{(1,0)} \right) \\ &= \left(\frac{\kappa}{2} \right)^4 i st A_{1234}^{(0)} \left(A_{1234,\mathcal{N}}^{(1,0)} + A_{1342,\mathcal{N}}^{(1,0)} + A_{1423,\mathcal{N}}^{(1,0)} \right) \end{aligned} \quad (2.11)$$

by replacing g^4 with $i(\kappa/2)^4$ and the color factors c_{ijkl} with n_{ijkl} in eq. (2.6). The $1/\epsilon^2$ poles of the planar amplitudes in eq. (2.11) cancel to give the expected $1/\epsilon$ leading IR divergence of one-loop supergravity amplitudes.

Similarly, two-loop four-point amplitudes of particles in the adjoint representation may be expressed as¹

$$\begin{aligned} \mathcal{A}_{\mathcal{N}}^{(2)}(1, 2, 3, 4) &= g^6 \left(c_{1234}^{(2P)} A_{1234,\mathcal{N}}^{(2P)} + c_{3421}^{(2P)} A_{3421,\mathcal{N}}^{(2P)} + c_{1234}^{(2NP)} A_{1234,\mathcal{N}}^{(2NP)} + c_{3421}^{(2NP)} A_{3421,\mathcal{N}}^{(2NP)} \right. \\ &\quad + c_{1342}^{(2P)} A_{1342,\mathcal{N}}^{(2P)} + c_{4231}^{(2P)} A_{4231,\mathcal{N}}^{(2P)} + c_{1342}^{(2NP)} A_{1342,\mathcal{N}}^{(2NP)} + c_{4231}^{(2NP)} A_{4231,\mathcal{N}}^{(2NP)} \\ &\quad \left. + c_{1423}^{(2P)} A_{1423,\mathcal{N}}^{(2P)} + c_{2341}^{(2P)} A_{2341,\mathcal{N}}^{(2P)} + c_{1423}^{(2NP)} A_{1423,\mathcal{N}}^{(2NP)} + c_{2341}^{(2NP)} A_{2341,\mathcal{N}}^{(2NP)} \right), \end{aligned} \quad (2.12)$$

where the basis of color factors is given by the two-loop planar (2P) and nonplanar (2NP) diagrams shown in fig. 1, namely,

$$c_{1234}^{(2P)} = \tilde{f}^{ea_1 b} \tilde{f}^{ba_2 c} \tilde{f}^{c g d} \tilde{f}^{d f e} \tilde{f}^{ga_3 h} \tilde{f}^{ha_4 f} \quad (2.13)$$

$$c_{1234}^{(2NP)} = \tilde{f}^{ea_1 b} \tilde{f}^{ba_2 c} \tilde{f}^{c g d} \tilde{f}^{h f e} \tilde{f}^{ga_3 h} \tilde{f}^{da_4 f} \quad (2.14)$$

The manifest symmetries $c_{1234} = c_{4321} = c_{2143} = c_{3412}$ of both planar and nonplanar color factors have been used to reduce the number of terms in eq. (2.12). Even taking these symmetries into account, this

¹Our convention for $A^{(2P)}$ and $A^{(2NP)}$ differs from ref. [7] by a sign.

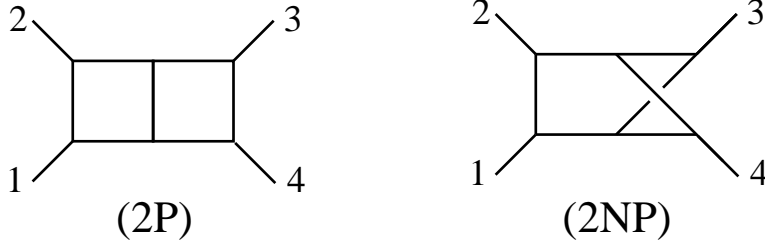


Figure 1: Two-loop planar and nonplanar diagrams

color basis is still over-complete: the nonplanar color factors satisfy $c_{1234}^{(2NP)} = c_{1243}^{(2NP)}$, and also can be rewritten in terms of planar ones

$$3c_{1234}^{(2NP)} = c_{1234}^{(2P)} - c_{2341}^{(2P)} - c_{1342}^{(2P)} + c_{3421}^{(2P)} \quad (2.15)$$

and further there exists a linear relation among planar color factors

$$0 = c_{1234}^{(2P)} - c_{2341}^{(2P)} + c_{1342}^{(2P)} - c_{3421}^{(2P)} + c_{1423}^{(2P)} - c_{4231}^{(2P)} \quad (2.16)$$

so there are only five independent two-loop color factors. Representations of amplitudes that manifest BCJ duality, however, often require the use of an overcomplete color basis.

Specializing to $\mathcal{N} = 4$ SYM, eq. (2.12) can be written in the form eq. (1.1) with [15, 22]

$$A_{1234, \mathcal{N}=4}^{(2P)} = -n_{1234}^{(2P)} \mathcal{I}_{1234}^{(2P)}, \quad A_{1234, \mathcal{N}=4}^{(2NP)} = -n_{1234}^{(2NP)} \mathcal{I}_{1234}^{(2NP)}, \quad (2.17)$$

where the two-loop scalar integrals associated with the diagrams in fig. 1 are

$$\begin{aligned} \mathcal{I}_{1234}^{(2P)} &= \int \frac{d^D p}{(2\pi)^D} \frac{d^D q}{(2\pi)^D} \frac{1}{p^2 (p-k_1)^2 (p-k_1-k_2)^2 (p+q)^2 q^2 (q-k_4)^2 (q-k_3-k_4)^2}, \\ \mathcal{I}_{1234}^{(2NP)} &= \int \frac{d^D p}{(2\pi)^D} \frac{d^D q}{(2\pi)^D} \frac{1}{p^2 (p-k_2)^2 (p+q)^2 (p+q+k_1)^2 q^2 (q-k_3)^2 (q-k_3-k_4)^2}, \end{aligned}$$

and the kinematic numerators (which are again independent of loop momenta) are

$$\begin{aligned} n_{1234}^{(2P)} &= n_{3421}^{(2P)} = n_{1234}^{(2NP)} = n_{3421}^{(2NP)} = s(stA_{1234}^{(0)}) \\ n_{1342}^{(2P)} &= n_{4231}^{(2P)} = n_{1342}^{(2NP)} = n_{4231}^{(2NP)} = u(stA_{1234}^{(0)}) \\ n_{1423}^{(2P)} &= n_{2341}^{(2P)} = n_{1423}^{(2NP)} = n_{2341}^{(2NP)} = t(stA_{1234}^{(0)}) \end{aligned} \quad (2.18)$$

The numerators (2.18) satisfy color-kinematic duality (see *e.g.* the discussion in ref. [7]), and so they can be used in a double-copy representation of the two-loop four-point amplitude of $\mathcal{N} + 4$ supergravity [7]

$$\begin{aligned} \mathcal{M}_{\mathcal{N}+4}^{(2)}(1, 2, 3, 4) &= i \left(\frac{\kappa}{2} \right)^6 stA_{1234}^{(0)} \left[s \left(A_{1234, \mathcal{N}}^{(2P)} + A_{3421, \mathcal{N}}^{(2P)} + A_{1234, \mathcal{N}}^{(2NP)} + A_{3421, \mathcal{N}}^{(2NP)} \right) \right. \\ &\quad + u \left(A_{1342, \mathcal{N}}^{(2P)} + A_{4231, \mathcal{N}}^{(2P)} + A_{1342, \mathcal{N}}^{(2NP)} + A_{4231, \mathcal{N}}^{(2NP)} \right) \\ &\quad \left. + t \left(A_{1423, \mathcal{N}}^{(2P)} + A_{2341, \mathcal{N}}^{(2P)} + A_{1423, \mathcal{N}}^{(2NP)} + A_{2341, \mathcal{N}}^{(2NP)} \right) \right] \end{aligned} \quad (2.19)$$

obtained by replacing g^6 with $i(\kappa/2)^6$ and the color factors $c_{ijkl}^{(2P, 2NP)}$ with $n_{ijkl}^{(2P, 2NP)}$ in eq. (2.12). The $1/\epsilon^4$ and $1/\epsilon^3$ divergences of the gauge theory amplitudes on the r. h. s. cancel to give the expected $1/\epsilon^2$ leading IR divergence for two-loop gravity amplitudes [7]. Moreover, it was verified in ref. [7] that the IR divergent part of eq. (2.19) is given by one-half of (the divergent part of) the square of the one-loop supergravity amplitude eq. (2.11).

2.3 Five-point amplitudes

One-loop five-point amplitudes of particles in the adjoint representation of a gauge theory may be decomposed in a basis of color factors of the pentagon diagram [21] as

$$\mathcal{A}_{\mathcal{N}}^{(1)}(1, 2, 3, 4, 5) = g^5 \sum_{S_5/\mathbb{Z}_5 \times \mathbb{Z}_2} c_{12345}^{(P)} A_{12345, \mathcal{N}}^{(1,0)} \quad (2.20)$$

where

$$c_{12345}^{(P)} = \tilde{f}^{ga_1b} \tilde{f}^{ba_2c} \tilde{f}^{ca_3d} \tilde{f}^{da_4e} \tilde{f}^{ea_5g} \quad (2.21)$$

is symmetric under cyclic permutations of 12345 and antisymmetric under $12345 \rightarrow 54321$. By expressing $c_{12345}^{(P)}$ in terms of the trace basis (2.2), one ascertains [18, 21] that the coefficients in the decomposition (2.20) are identical to the planar one-loop color-ordered amplitudes.

Carrasco and Johansson [5] showed that a representation of the $\mathcal{N} = 4$ SYM five-point amplitude satisfying color-kinematic duality requires the use of an overcomplete basis of color factors, which includes the box-plus-line color diagram

$$c_{12;345}^{(B)} = \tilde{f}^{a_1a_2b} \tilde{f}^{bcg} \tilde{f}^{ca_3d} \tilde{f}^{da_4e} \tilde{f}^{ea_5g} \quad (2.22)$$

in addition to the pentagon color factors $c_{12345}^{(P)}$. Then

$$\mathcal{A}_{\mathcal{N}=4}^{(1)}(1, 2, 3, 4, 5) = g^5 \sum_{S_5} \left(\frac{1}{10} i \beta_{12345} c_{12345}^{(P)} \mathcal{I}_{12345}^{(P)} + \frac{1}{4} i \gamma_{12;345} c_{12;345}^{(B)} \mathcal{I}_{12;345}^{(B)} \right) \quad (2.23)$$

where explicit expressions for the numerators β_{12345} and $\gamma_{12;345}$ as well as the integrals $\mathcal{I}_{12345}^{(P)}$ and $\mathcal{I}_{12;345}^{(B)}$ are found in ref. [5]. For the purposes of sec. 3, we only need that β_{12345} is symmetric under cyclic permutations of its indices and antisymmetric under reversal of the indices, and additionally satisfies the property [5]

$$\beta_{ijklm} - \beta_{jiklm} - \beta_{ijlkm} + \beta_{jilkm} = 0. \quad (2.24)$$

Because the kinematic numerators satisfy color-kinematic duality, one may use them to obtain a double-copy representation of the two-loop five-point amplitude of $\mathcal{N} + 4$ supergravity [6]

$$\mathcal{M}_{\mathcal{N}+4}^{(1)}(1, 2, 3, 4, 5) = \left(\frac{\kappa}{2} \right)^5 \sum_{S_5/(\mathbb{Z}_5 \times \mathbb{Z}_2)} i \beta_{12345} A_{12345, \mathcal{N}}^{(1,0)} \quad (2.25)$$

by replacing g^5 with $i(\kappa/2)^5$ and the color factors $c_{ijklm}^{(P)}$ with β_{ijklm} in eq. (2.20).

3 Relations between supergravity and most-subleading-color SYM amplitudes

In the previous section, we reviewed various expressions for $\mathcal{N} \geq 4$ supergravity amplitudes in which the expected leading $\mathcal{O}(1/\epsilon^L)$ IR divergence emerges after cancellation of the higher-order IR divergences of planar gauge theory amplitudes. The IR divergences of gauge theory amplitudes that are subleading in the $1/N$ expansion are less severe than those of planar amplitudes [16, 17, 23], as we will review in sec. 4, with the most-subleading-color amplitude having a leading divergence that is only $\mathcal{O}(1/\epsilon^L)$. This suggests the possibility of linear relations between supergravity amplitudes and the most-subleading-color amplitudes of gauge theories used in the double copy representation. In this section, we will exhibit such relations for one- and two-loop four-point functions and the one-loop five-point function of $\mathcal{N} \geq 4$ supergravity. In all these cases, the kinematic numerators of $\mathcal{N} = 4$ SYM theory are independent of the loop momenta. Whether such relations exist more generally remains an open question.

3.1 Four-point amplitudes

By expressing the four-point one-loop color factor (2.7) in terms of the trace basis (2.3), one shows that the one-loop subleading-color amplitudes are given by

$$A_{13;42,\mathcal{N}}^{(1,1)} = A_{14;23,\mathcal{N}}^{(1,1)} = A_{12;34,\mathcal{N}}^{(1,1)} = 2 \left(A_{1234,\mathcal{N}}^{(1,0)} + A_{1342,\mathcal{N}}^{(1,0)} + A_{1423,\mathcal{N}}^{(1,0)} \right) \quad (3.1)$$

which is just the one-loop $U(1)$ decoupling relation [19]. The relation (3.1) may be used to recast the one-loop $\mathcal{N} + 4$ supergravity amplitude (2.11) as

$$\mathcal{M}_{\mathcal{N}+4}^{(1)}(1, 2, 3, 4) = \frac{1}{2} \left(\frac{\kappa}{2} \right)^4 i s t A_{1234}^{(0)} A_{12;34,\mathcal{N}}^{(1,1)} \quad (3.2)$$

in terms of the one-loop subleading-color amplitude of \mathcal{N} SYM theory; both sides have a leading $1/\epsilon$ IR divergence. This generalizes the relation for the one-loop $\mathcal{N} = 8$ supergravity amplitude in ref. [16].

Similarly, one may expand the two-loop color factors (2.13) and (2.14) in eq. (2.12) in terms of the trace basis (2.3), (2.4) to obtain [15]

$$A_{1234,\mathcal{N}}^{(2,0)} = A_{1234,\mathcal{N}}^{(2P)} + A_{2341,\mathcal{N}}^{(2P)} \quad (3.3)$$

$$\begin{aligned} A_{12;34,\mathcal{N}}^{(2,1)} = & 2 \left(3A_{1234,\mathcal{N}}^{(2P)} + 3A_{3421,\mathcal{N}}^{(2P)} + 2A_{1234,\mathcal{N}}^{(2NP)} + 2A_{3421,\mathcal{N}}^{(2NP)} \right. \\ & \left. - A_{1342,\mathcal{N}}^{(2NP)} - A_{4231,\mathcal{N}}^{(2NP)} - A_{1423,\mathcal{N}}^{(2NP)} - A_{2341,\mathcal{N}}^{(2NP)} \right) \end{aligned} \quad (3.4)$$

$$\begin{aligned} A_{1234,\mathcal{N}}^{(2,2)} = & 2 \left(A_{1234,\mathcal{N}}^{(2P)} + A_{3421,\mathcal{N}}^{(2P)} + A_{1234,\mathcal{N}}^{(2NP)} + A_{3421,\mathcal{N}}^{(2NP)} \right. \\ & - 2A_{1342,\mathcal{N}}^{(2P)} - 2A_{4231,\mathcal{N}}^{(2P)} - 2A_{1342,\mathcal{N}}^{(2NP)} - 2A_{4231,\mathcal{N}}^{(2NP)} \\ & \left. + A_{1423,\mathcal{N}}^{(2P)} + A_{2341,\mathcal{N}}^{(2P)} + A_{1423,\mathcal{N}}^{(2NP)} + A_{2341,\mathcal{N}}^{(2NP)} \right). \end{aligned} \quad (3.5)$$

As we show in sec. 4, the leading IR pole of the amplitude $A^{(L,k)}$ goes as $\mathcal{O}(1/\epsilon^{2L-k})$, and so the leading IR divergence of $A^{(2,2)}$ matches that of the two-loop supergravity amplitude. This is reflected in the

following expression for the two-loop $\mathcal{N} + 4$ supergravity amplitude

$$\mathcal{M}_{\mathcal{N}+4}^{(2)}(1, 2, 3, 4) = -\frac{1}{6}\left(\frac{\kappa}{2}\right)^6 ist A_{1234}^{(0)} \left[u A_{1234, \mathcal{N}}^{(2,2)} + t A_{1342, \mathcal{N}}^{(2,2)} + s A_{1423, \mathcal{N}}^{(2,2)} \right] \quad (3.6)$$

which is easily verified by substituting eq. (3.5) into eq. (3.6) and comparing with eq. (2.19). Equation (3.6) generalizes the result in ref. [16] expressing the two-loop $\mathcal{N} = 8$ supergravity amplitude in terms of subleading-color $\mathcal{N} = 4$ SYM amplitudes.

Both one-loop (3.2) and two-loop (3.6) relations may be written in a uniform way

$$\left(\sqrt{2}g^2N\right)^L M_{SG, \mathcal{N}+4}^{(L)}(s, t) = \frac{1}{3} \left[\left(\left(\frac{\kappa}{2}\right)^2 u\right)^L M_{SYM, \mathcal{N}}^{(L,L)}(s, t) + \text{cyclic perms of } s, t, u \right] \quad (3.7)$$

valid for $L = 0, 1, 2$ and for $\mathcal{N} = 0, 1, 2, 4$, in terms of the ratios (2.5) of \mathcal{N} SYM amplitudes and the ratio of loop-level to tree-level $\mathcal{N} + 4$ supergravity amplitudes

$$M_{SG, \mathcal{N}+4}^{(L)}(s, t) \equiv \frac{\mathcal{M}_{\mathcal{N}+4}^{(L)}(1, 2, 3, 4)}{\mathcal{M}^{(0)}(1, 2, 3, 4)} \quad (3.8)$$

where the tree-level supergravity amplitude is given by the KLT relation

$$\mathcal{M}^{(0)}(1, 2, 3, 4) = -\frac{ist}{u} [A_{1234}^{(0)}]^2 \quad (3.9)$$

independent of the number of supersymmetries. This form emphasizes that the most subleading SYM ratio $M_{SYM}^{(L,L)}$ is replaced by the supergravity ratio $M_{SG}^{(L)}$, and the dimensionless coupling of SYM g^2N is replaced by the effective dimensionless supergravity coupling $(\kappa/2)^2 u$ (where the kinematic variable u must appear for dimensional reasons). Note that the simple form (3.7) is not valid for $L > 2$ (though perhaps some modification thereof could be).

3.2 Five-point amplitudes

By evaluating the pentagon diagram color factor (2.21) in terms of the trace basis (2.2), one shows that the subleading-color amplitudes satisfy

$$\begin{aligned} A_{12;345, \mathcal{N}}^{(1,1)} &= A_{12345, \mathcal{N}}^{(1,0)} + A_{23415, \mathcal{N}}^{(1,0)} + A_{13425, \mathcal{N}}^{(1,0)} + A_{34215, \mathcal{N}}^{(1,0)} + A_{32415, \mathcal{N}}^{(1,0)} + A_{13245, \mathcal{N}}^{(1,0)} \\ &\quad + A_{21345, \mathcal{N}}^{(1,0)} + A_{23145, \mathcal{N}}^{(1,0)} + A_{31425, \mathcal{N}}^{(1,0)} + A_{34125, \mathcal{N}}^{(1,0)} + A_{32145, \mathcal{N}}^{(1,0)} + A_{31245, \mathcal{N}}^{(1,0)} \end{aligned} \quad (3.10)$$

the relations written long ago in refs. [19, 24]. In sec. 4 we verify that the $1/\epsilon^2$ poles on the r. h. s. of this equation cancel, leaving only an $\mathcal{O}(1/\epsilon)$ pole for $A^{(1,1)}$.

Next, we will show that the following SYM-supergravity relation

$$\mathcal{M}_{\mathcal{N}+4}^{(1)}(1, 2, 3, 4, 5) = \frac{1}{20} \left(\frac{\kappa}{2}\right)^5 \sum_{S_5} i\beta_{12345} A_{12;345, \mathcal{N}}^{(1,1)} \quad (3.11)$$

which was proven in ref. [18] for $\mathcal{N} = 4$, remains valid for $\mathcal{N} \leq 4$. First, we use eq. (3.10) to rewrite eq. (3.11) as

$$\mathcal{M}_{\mathcal{N}+4}^{(1)}(1, 2, 3, 4, 5) = \frac{1}{20} \left(\frac{\kappa}{2}\right)^5 \sum_{S_5} i\delta_{12345} A_{12345, \mathcal{N}}^{(1,0)} \quad (3.12)$$

where

$$\begin{aligned} \delta_{12345} = & \beta_{12345} + \beta_{41235} + \beta_{14235} + \beta_{43125} + \beta_{42135} + \beta_{13245} \\ & + \beta_{21345} + \beta_{31245} + \beta_{24135} + \beta_{34125} + \beta_{32145} + \beta_{23145} \end{aligned} \quad (3.13)$$

Here we have relabeled the sums over S_5 to give $A_{12345, \mathcal{N}}^{(1,0)}$ as a common factor rather than β_{12345} . We use the properties of β_{12345} under cyclic permutation and reversal of indices to rewrite δ_{12345} as

$$\delta_{12345} = \beta_{12345} - \beta_{21435} + \beta_{14235} - \beta_{41325} + \beta_{13245} - \beta_{31425} \quad (3.14)$$

Then we apply the property (2.24) which holds for the kinematic numerator of the $\mathcal{N} = 4$ five-point amplitude to obtain

$$\delta_{12345} = 2\beta_{12345} \quad (3.15)$$

Substituting this into eq. (3.12), we see that eq. (3.11) is equivalent to eq. (2.25). Each of the terms on the r.h.s. of eq. (3.11) has a leading $\mathcal{O}(1/\epsilon)$ divergence.

Therefore we again find a relation between supergravity and the most-subleading-color SYM amplitude. Note that its derivation depends not only on the group theory relation (3.10), but also on properties (2.24) that are valid because β_{12345} are the kinematic numerators of the $\mathcal{N} = 4$ SYM theory, which satisfy BCJ duality.

4 IR divergences of subleading-color amplitudes

In refs. [16, 17, 23] the structure of IR divergences of subleading-color amplitudes of $\mathcal{N} = 4$ theory was explored. In this section, we will see that some of these properties carry over to $\mathcal{N} < 4$ SYM theory.

4.1 Four-point amplitudes

We wish to show that the *leading* IR singularities of the subleading-color amplitudes for four-point functions of particles in the adjoint representation are the same in a generic $SU(N)$ gauge theory as in $\mathcal{N} = 4$ SYM theory [16, 17]. In particular, the color-ordered amplitude $A^{(L,k)}$ has leading pole of $\mathcal{O}(1/\epsilon^{2L-k})$, whose coefficient we will specify below. Subleading IR poles, however, will differ for different gauge theories due to the running of the coupling constant and differing anomalous dimensions.

It is convenient [25, 26] to organize color-ordered amplitudes into a vector²

$$|A\rangle = (A_{1234}, A_{1342}, A_{1423}, A_{13;42}, A_{14;23}, A_{12;34}) . \quad (4.1)$$

²We adopt here the basis used in ref. [27], which differs slightly from than that used in refs. [17, 28]. Also, note that the overall factor of g^2 for a four-point function $\mathcal{A}(1, 2, 3, 4)$ has been stripped off.

We follow refs. [29, 30] by organizing the IR divergences as

$$\left| A \left(\frac{s_{ij}}{\mu^2}, a(\mu^2), \epsilon \right) \right\rangle = J \left(\frac{Q^2}{\mu^2}, a(\mu^2), \epsilon \right) \mathbf{S} \left(\frac{s_{ij}}{Q^2}, \frac{Q^2}{\mu^2}, a(\mu^2), \epsilon \right) \left| H \left(\frac{s_{ij}}{Q^2}, \frac{Q^2}{\mu^2}, a(\mu^2), \epsilon \right) \right\rangle \quad (4.2)$$

where the prefactors J (“jet function”) and \mathbf{S} (“soft function”) characterize the long-distance IR-divergent behavior, while the short-distance behavior of the amplitude is characterized by $|H\rangle$ (“hard function”), and is finite as $\epsilon \rightarrow 0$. (Quantities in boldface act as matrices on the color space vectors.) In eq. (4.2), $s_{ij} = (k_i + k_j)^2$, μ is the renormalization scale,

$$a(\mu^2) = \frac{g^2(\mu^2)N}{8\pi^2} (4\pi e^{-\gamma})^\epsilon \quad (4.3)$$

and Q is an arbitrary factorization scale. The amplitude is independent of the factorization scale, but its split into J , \mathbf{S} , and H depends on Q .

For $\mathcal{N} = 4$ SYM theory, which has $\beta_0 = 0$, $\log J$ has only $1/\epsilon^2$ and $1/\epsilon$ poles, whose coefficients are given by anomalous dimensions. For generic gauge theories, the coupling constant runs, and so $\log J$ will also have higher-order poles, starting at two-loop order in a . These poles, whose coefficients depend on β_0 , go up through $1/\epsilon^{L+1}$ at L -loop order [30]. The one-loop $\mathcal{O}(a/\epsilon^2)$ contribution to $\log J$, however, produces the leading L -loop IR divergence of $\mathcal{O}(a^L/\epsilon^{2L})$ in J , because the β_0 -dependent contributions of $\mathcal{O}(as^L/\epsilon^{L+1})$ are subleading in the $1/\epsilon$ expansion.

The soft function [29, 30]

$$\mathbf{S} \left(\frac{s_{ij}}{Q^2}, \frac{Q^2}{\mu^2}, a(\mu^2), \epsilon \right) = \text{P exp} \left[-\frac{1}{2} \int_0^{Q^2} \frac{d\tilde{\mu}^2}{\tilde{\mu}^2} \mathbf{\Gamma} \left(\frac{s_{ij}}{Q^2}, \bar{a} \left(\frac{\mu^2}{\tilde{\mu}^2}, a(\mu^2), \epsilon \right) \right) \right] \quad (4.4)$$

depends on the anomalous dimension matrix

$$\mathbf{\Gamma} \left(\frac{s_{ij}}{Q^2}, a(\mu^2) \right) = \sum_{\ell=1}^{\infty} a(\mu^2)^\ell \mathbf{\Gamma}^{(\ell)} \left(\frac{s_{ij}}{Q^2} \right), \quad (4.5)$$

where the leading form of the running coupling is given by [29, 30]

$$\bar{a} \left(\frac{\mu^2}{\tilde{\mu}^2}, a(\mu^2), \epsilon \right) = a(\mu^2) \left(\frac{\mu^2}{\tilde{\mu}^2} \right)^\epsilon \sum_{n=0}^{\infty} \left[\frac{\beta_0}{4\pi\epsilon} \left(\left(\frac{\mu^2}{\tilde{\mu}^2} \right)^\epsilon - 1 \right) a(\mu^2) \right]^n \quad (4.6)$$

If the matrices $\mathbf{\Gamma}^{(\ell)}$ all commute with one another (which occurs if they are proportional to $\mathbf{\Gamma}^{(1)}$, which is true through at least two loops [30]) we can eliminate the path ordering of the exponential and integrate the terms to obtain

$$\mathbf{S} \left(\frac{s_{ij}}{Q^2}, \frac{Q^2}{\mu^2}, a(\mu^2), \epsilon \right) = \exp \left[\frac{1}{2} \sum_{\ell=1}^{\infty} a(\mu^2)^\ell \left(\frac{\mu^2}{Q^2} \right)^{\ell\epsilon} \frac{\mathbf{\Gamma}^{(\ell)}}{\ell\epsilon} \left(1 + \mathcal{O} \left(\frac{a(\mu^2)}{\epsilon} \right) \right) \right]. \quad (4.7)$$

The omitted terms, which depend on β_0 , will be subleading in the $1/\epsilon$ expansion relative to the $\mathbf{\Gamma}^{(\ell)}/\epsilon$ terms in $\log \mathbf{S}$. Finally, when we exponentiate to obtain \mathbf{S} , we see that the contribution of $\mathbf{\Gamma}^{(\ell)}/\epsilon$ will

be subleading in the $1/\epsilon$ expansion relative to the one-loop contribution $\mathbf{\Gamma}^{(1)}/\epsilon$, as in the case of the jet function.

To summarize, we find, as in the case of $\mathcal{N} = 4$ SYM theory [17], that the leading IR poles of the amplitude arise from the exponentiation of the one-loop divergences³

$$|A\rangle \Big|_{\text{most divergent}} = \exp \left(-\frac{g^2 N}{4\pi^2} \left[\frac{1}{\epsilon^2} \mathbb{1} - \frac{1}{4\epsilon} \mathbf{\Gamma}^{(1)} \right] \right) |A^{(0)}\rangle \quad (4.8)$$

We now decompose $|A\rangle$ in a loop and $1/N$ expansion

$$|A\rangle = \sum_{L=0}^{\infty} \sum_{k=0}^L g^{2L} N^{L-k} |A^{(L,k)}\rangle \quad (4.9)$$

From eq. (4.8), the leading IR pole of the planar amplitude is simply

$$|A^{(L,0)}\rangle = \frac{1}{L!} \left(-\frac{1}{4\pi^2} \right)^L \frac{1}{\epsilon^{2L}} |A^{(0)}\rangle + \mathcal{O} \left(\frac{1}{\epsilon^{2L-1}} \right) \quad (4.10)$$

The one-loop anomalous dimension matrix is given by

$$\mathbf{\Gamma}^{(1)} = \frac{1}{N} \sum_{i=1}^4 \sum_{j \neq i}^4 \mathbf{T}_i \cdot \mathbf{T}_j \log \left(\frac{\mu^2}{-s_{ij}} \right) \quad (4.11)$$

where \mathbf{T}_i are the $\text{SU}(N)$ generators in the adjoint representation. In the basis defined in eq. (4.1), the anomalous dimension matrix takes the form

$$\mathbf{\Gamma}^{(1)} = 2 \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} + \frac{2}{N} \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \quad (4.12)$$

with

$$b = \begin{pmatrix} 0 & -Y & X \\ Z & 0 & -X \\ -Z & Y & 0 \end{pmatrix} \quad c = \begin{pmatrix} 0 & -2X & 2Y \\ 2X & 0 & -2Z \\ -2Y & 2Z & 0 \end{pmatrix} \quad (4.13)$$

and

$$X = \log \left(\frac{t}{u} \right), \quad Y = \log \left(\frac{u}{s} \right), \quad Z = \log \left(\frac{s}{t} \right). \quad (4.14)$$

and the matrices a and d will not be needed. From eq. (4.8), we can read off the coefficient of the leading IR pole for the subleading-color amplitudes

$$|A^{(L,k)}\rangle = \frac{1}{k!(L-k)!(-4\pi^2)^L(-2)^k \epsilon^{2L-k}} \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}^k |A^{(0)}\rangle + \mathcal{O} \left(\frac{1}{\epsilon^{2L-k-1}} \right). \quad (4.15)$$

³At this point, to simplify the formulas, we set the factorization scale Q equal to the renormalization scale μ .

We can write this explicitly as

$$\left. \begin{matrix} A_{13;42}^{(L,2m+1)} \\ A_{14;23}^{(L,2m+1)} \\ A_{12;34}^{(L,2m+1)} \end{matrix} \right\} = \left[\frac{(-1)^{L-1} (X^2 + Y^2 + Z^2)^m (YA_{1423}^{(0)} - XA_{1342}^{(0)})}{(2m+1)!(L-2m-1)!(4\pi^2)^L 2^m} \right] \frac{1}{\epsilon^{2L-2m-1}} + \mathcal{O}\left(\frac{1}{\epsilon^{2L-2m-2}}\right) \quad (4.16)$$

and

$$\left. \begin{matrix} A_{1234}^{(L,2m+2)} \\ A_{1342}^{(L,2m+2)} \\ A_{1423}^{(L,2m+2)} \end{matrix} \right\} = \frac{(-1)^L (X^2 + Y^2 + Z^2)^m (YA_{1423}^{(0)} - XA_{1342}^{(0)})}{(2m+2)!(L-2m-2)!(4\pi^2)^L 2^{m+1}} \left\{ \begin{matrix} X - Y \\ Z - X \\ Y - Z \end{matrix} \right\} \frac{1}{\epsilon^{2L-2m-2}} + \mathcal{O}\left(\frac{1}{\epsilon^{2L-2m-3}}\right) \quad (4.17)$$

where we observe that the term in the numerator is invariant under cyclic permutations of s , t , and u :

$$YA_{1423}^{(0)} - XA_{1342}^{(0)} = ZA_{1342}^{(0)} - YA_{1234}^{(0)} = XA_{1234}^{(0)} - ZA_{1423}^{(0)} \quad (4.18)$$

as a consequence of the tree-level BCJ relations [1]

$$tA_{1423}^{(0)} = sA_{1342}^{(0)}, \quad uA_{1342}^{(0)} = tA_{1234}^{(0)}, \quad sA_{1234}^{(0)} = uA_{1423}^{(0)}. \quad (4.19)$$

As expected, eqs. (4.16) and (4.17) are consistent with the group theory constraints on color-ordered four-point amplitudes derived in ref. [27].

4.2 Five-point amplitudes and generalization to all $A_{n;j}$

In ref. [23], it was shown that the leading IR divergence of the subleading-color one-loop n -point amplitude is of $\mathcal{O}(1/\epsilon)$ for $\mathcal{N} = 4$ SYM theory. Here we show that the same result applies to generic $SU(N)$ gauge theories.

First consider five-point functions. We recall from eq. (3.10) that the one-loop subleading-color five-point amplitude obeys the relation

$$\begin{aligned} A_{12;345}^{(1,1)} &= A_{12345}^{(1,0)} + A_{23415}^{(1,0)} + A_{13425}^{(1,0)} + A_{34215}^{(1,0)} + A_{32415}^{(1,0)} + A_{13245}^{(1,0)} \\ &\quad + A_{21345}^{(1,0)} + A_{23145}^{(1,0)} + A_{31425}^{(1,0)} + A_{34125}^{(1,0)} + A_{32145}^{(1,0)} + A_{31245}^{(1,0)} \end{aligned} \quad (4.20)$$

purely as a result of group theory. The planar one-loop five-point amplitude can be written

$$A_{12345}^{(1,0)} = A_{12345}^{(0)} M_{12345} \quad (4.21)$$

For $\mathcal{N} = 4$ SYM theory, M_{12345} is helicity-independent. For generic $SU(N)$ gauge theories, this is no longer the case, except for the IR-divergent contribution [31]. In particular

$$M_{12345} = -\left(\frac{5}{16\pi^2}\right) \frac{1}{\epsilon^2} + \mathcal{O}\left(\frac{1}{\epsilon}\right) \quad (4.22)$$

Furthermore there are six relations among the twelve tree-level five-point amplitudes $A_{12345}^{(0)}$, known as Kleiss-Kuijf relations [32], here shown for n -point tree amplitudes

$$A^{(0)}(1, \{\alpha\}, n, \{\beta\}) = (-1)^{n_\beta} \sum_{\{\sigma\}_i \in OP(\{\alpha\}, \{\beta^T\})} A^{(0)}(1, \{\sigma\}_i, n) \quad (4.23)$$

It was shown in ref. [23] that these can be used to rewrite eqs. (4.20) and (4.21) as

$$\begin{aligned} A_{45;123}^{(1,1)} &= A_{12345}^{(1,0)} [(M_{12345} - M_{41235}) + (M_{43125} - M_{31245})] \\ &+ A_{12435}^{(1,0)} [(M_{12435} - M_{31245}) + (M_{34125} - M_{41235})] \\ &+ A_{14235}^{(1,0)} [(M_{14235} - M_{31425}) + (M_{34125} - M_{41235})] \\ &+ A_{13245}^{(1,0)} [(M_{23145} - M_{42315}) + (M_{43125} - M_{31245})] \\ &+ A_{13425}^{(1,0)} [(M_{23145} - M_{31425}) + (M_{43125} - M_{24315})] \\ &+ A_{14325}^{(1,0)} [(M_{23145} - M_{31425}) + (M_{34125} - M_{23415})] \end{aligned} \quad (4.24)$$

Due to the alternating signs in eq. (4.24), the leading $\mathcal{O}(1/\epsilon^2)$ pole cancels out, and $A_{12;345}^{(1,1)}$ goes as $\mathcal{O}(1/\epsilon)$.

The same conclusion can be drawn for arbitrary n -point functions of particles in the adjoint representation. Moreover, in that case, as we saw in (2.1) we have a sum over double trace structures, multiplying color-ordered amplitudes $A_{n;j}$, and the same conclusion applies for all the $A_{n;j}$. Since the ingredients of the proof rely only on group theory, together with the fact that the leading IR divergence of the planar one-loop amplitude is proportional to $1/\epsilon^2$ times the tree-level amplitude, the alternating signs [23] in the one-loop decoupling relation guarantee that the leading $1/\epsilon^2$ pole cancels out of all the $A_{n;j}$ amplitudes of a generic $SU(N)$ gauge theory, generalizing the result of ref. [23].

4.3 Consistency of IR divergences for the five-point amplitude

In this section we study the consequences of the relations in sec. 3 for the IR divergences of \mathcal{N} SYM, and the consistency conditions associated with them.

In previous work by two of the authors [33], it was shown that in order to have a linear relation between the subleading-color five-point amplitude of $\mathcal{N} = 4$ SYM and the five-point amplitude of $\mathcal{N} = 8$ supergravity

$$\mathcal{M}_{\mathcal{N}=8}^{(1)}(1, 2, 3, 4, 5) = \sum_{(ij)} \beta_{(ij)} A_{ij;fgh,\mathcal{N}}^{(1,1)} \quad (4.25)$$

where (ij) are pairs of (different) indices, and $(ijfgh)$ is a permutation of (12345), the IR divergences must be the same, which gives

$$\frac{1}{\epsilon^2} \mathcal{M}^{(0)}(1, 2, 3, 4, 5) \sum_{i < j} s_{ij} (-s_{ij})^{-\epsilon} = \frac{r_\Gamma}{\epsilon^2} \sum_{(fg) \neq lmn} \beta_{(fg)} \sum_{i < j} (-s_{ij})^{-\epsilon} \sum_{abc \in S_3} \epsilon_{lmn} [A_{ijabc}^{(0)}], \quad (4.26)$$

where the sum is over pairs (fg) such that $(fglmn)$ is a permutation of (12345) , and the tree supergravity and SYM amplitudes are the same for all \mathcal{N} , and $\epsilon_{lmn}[A_{ijabc}^{(0)}]$ means we multiply the amplitude by the sign of the permutation l, m, n inside i, j, a, b, c . In turn, for this relation to be true, the coefficients $\beta_{(ij)}$ must satisfy the relation

$$\sum_{(fg)} N_{(ij),(fg)} \beta_{(fg)} = \mathcal{M}^{(0)}(1, 2, 3, 4, 5) s_{ij} \quad (4.27)$$

where

$$N_{(ij),(fg)} = \sum_{abc \in S_3} \epsilon_{lmn} [A_{ijabc}^{(0)}] \quad (4.28)$$

is a matrix of rank at most 9, since $\sum_{(ij)} N_{(ij),(fg)} = 0$. Since the notation is a bit dense, we write a few examples of components for clarity

$$\begin{aligned} N_{(12),(12)} &= \sum_{perm.of(345)} \epsilon_{345} [A_{12(345)}^{(0)}] = A_{12345}^{(0)} - A_{12543}^{(0)} - A_{12435}^{(0)} + A_{12534}^{(0)} - A_{12354}^{(0)} + A_{12453}^{(0)} \\ N_{(12),(13)} &= \sum_{perm.of(345)} \epsilon_{245} [A_{12(345)}^{(0)}] = A_{12345}^{(0)} - A_{12543}^{(0)} + A_{12435}^{(0)} - A_{12534}^{(0)} - A_{12354}^{(0)} + A_{12453}^{(0)} \\ N_{(13),(13)} &= \sum_{perm.of(245)} \epsilon_{245} [A_{13(245)}^{(0)}] = A_{13245}^{(0)} - A_{13542}^{(0)} - A_{13425}^{(0)} + A_{13524}^{(0)} - A_{13254}^{(0)} + A_{13452}^{(0)} \end{aligned} \quad (4.29)$$

Since by symmetry we only need to prove (4.27) for a single set of (ij) , the matching of the IR divergences implies

$$\mathcal{M}^{(0)}(1, 2, 3, 4, 5) s_{12} = \sum_{(fg)} N_{(12),(fg)} \beta_{(fg)} = \sum_{(fg)} \sum_{abc \in S_3} \epsilon_{lmn} [A_{12abc}^{(0)}] \beta_{(fg)} \quad (4.30)$$

In ref. [33], the explicit solution for $\beta_{(fg)}$ was not known.

We now show that the coefficients $\beta_{(fg)}$ arising from the relation of the previous section do satisfy this relation. As shown in [18], we can rewrite eq. (3.11) as

$$\mathcal{M}_{\mathcal{N}=8}^{(1)}(1, 2, 3, 4, 5) = i \left(\frac{\kappa}{2} \right)^5 \sum_{S_5/\mathbb{Z}_2 \times S_3} \frac{\gamma_{34} + \gamma_{45} + \gamma_{53}}{10} A_{12;345,\mathcal{N}=4}^{(1,1)} \quad (4.31)$$

where

$$\gamma_{ij} = \beta_{ijklm} - \beta_{jiklm} \quad (4.32)$$

are independent of the order of k, l, m , are antisymmetric, $\gamma_{ij} = -\gamma_{ji}$ and satisfy

$$\sum_{i=1}^5 \gamma_{ij} = 0 \quad (4.33)$$

which means we have a basis of only 6 independent γ_{ij} 's, which we can take to be $\gamma_{12}, \gamma_{13}, \gamma_{14}, \gamma_{23}, \gamma_{24}, \gamma_{34}$. We then obtain, by identifying with (4.25) that the $\beta_{(ij)}$ coefficients are given by

$$\beta_{(12)} = \frac{i}{10} (\gamma_{34} + \gamma_{45} + \gamma_{53}) \quad (4.34)$$

and permutations. In the permutations, we have to be careful of the order in the sum, since $\gamma_{ij} = -\gamma_{ji}$. We take it to be the cyclic order of the remainder of the indices in the set 1, 2, 3, 4, 5, since this is also the convention taken for the matrix $N_{(ij),(fg)}$.

Expanding the right hand side of (4.30) and substituting the coefficients from (4.34), and writing only in terms of the 6 independent γ_{ij} basis members, we obtain after some algebra

$$\begin{aligned} \frac{i}{10} \Big\{ & 5A_{12345}^{(0)}[\gamma_{12} + \gamma_{13} + \gamma_{14} + \gamma_{23} + \gamma_{24} + \gamma_{34}] \\ & + 5A_{12435}^{(0)}[\gamma_{12} + \gamma_{13} + \gamma_{14} + \gamma_{23} + \gamma_{24} - \gamma_{34}] \\ & + 5A_{12453}^{(0)}[\gamma_{12} - \gamma_{13} + \gamma_{14} - \gamma_{23} + \gamma_{24} + \gamma_{34}] \\ & + 5A_{12543}^{(0)}[\gamma_{12} - \gamma_{13} - \gamma_{14} - \gamma_{23} - \gamma_{24} - \gamma_{34}] \\ & + 5A_{12534}^{(0)}[\gamma_{12} - \gamma_{13} - \gamma_{14} - \gamma_{23} - \gamma_{24} + \gamma_{34}] \\ & + 5A_{12354}^{(0)}[\gamma_{12} + \gamma_{13} - \gamma_{14} + \gamma_{23} - \gamma_{24} - \gamma_{34}] \Big\} \end{aligned} \quad (4.35)$$

which can then be finally written as

$$\begin{aligned} \mathcal{M}^{(0)}(1, 2, 3, 4, 5)_{s_{12}} &= i \Big[A_{12345}^{(0)}\beta_{12345} + A_{12435}^{(0)}\beta_{12435} + A_{12543}^{(0)}\beta_{12543} \\ &+ A_{12534}^{(0)}\beta_{12534} + A_{12453}^{(0)}\beta_{12453} + A_{12354}^{(0)}\beta_{12354} \Big] \end{aligned} \quad (4.36)$$

or more generally

$$\mathcal{M}^{(0)}(i, j, a, b, c)_{s_{ij}} = \sum_{\sigma \in S_3} i \beta_{ij\sigma(a)\sigma(b)\sigma(c)} A_{ij\sigma(a)\sigma(b)\sigma(c)}^{(0)} \quad (4.37)$$

This is a kind of tree level version of the BCJ relation (color-kinematic duality). However, note that the numerators are for *one-loop* amplitudes, yet they are used to multiply *tree amplitudes*. This relation can be proven as follows.

We take a specific helicity configuration, namely $(1^- 2^- 3^+ 4^+ 5^+)$. Then we have

$$\begin{aligned} \beta_{12345} &= \langle 12 \rangle^4 \frac{[12][23][34][45][51]}{\epsilon(1234)} \\ A_{12345}^{(0)} &= \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle} \\ \mathcal{M}^{(0)}(1, 2, 3, 4, 5) &= i \frac{\langle 12 \rangle^8 \epsilon(1234)}{N(5)} \\ N(5) &= \prod_{i < j} \langle ij \rangle \end{aligned} \quad (4.38)$$

and where the symbol $\epsilon(1234)$ satisfies

$$\epsilon(1234) = 4i\epsilon_{\mu\nu\rho\sigma} k_1^\mu k_2^\nu k_3^\rho k_4^\sigma = [12]\langle 23 \rangle [34]\langle 41 \rangle - \langle 12 \rangle [23] \langle 34 \rangle [41], \quad (4.39)$$

is totally antisymmetric, and satisfies $\epsilon(1234) = -\epsilon(1235)$, etc., as can be easily checked.

We can then expand the right hand side of (4.36), and using the properties and form of $\epsilon(1234)$ we get after some algebra

$$i \frac{\langle 12 \rangle^8 [12]}{N(5)} \left[\langle 14 \rangle \langle 24 \rangle \langle 35 \rangle [34] [45] + \langle 13 \rangle \langle 23 \rangle \langle 45 \rangle [34] [35] + \langle 15 \rangle \langle 25 \rangle \langle 34 \rangle [35] [45] \right] \quad (4.40)$$

Using the helicity spinor properties

$$\langle ij \rangle \langle kl \rangle = \langle ik \rangle \langle jl \rangle + \langle il \rangle \langle kj \rangle, \quad \sum_{i \neq j, k} \langle ji \rangle [ik] = 0 \quad (4.41)$$

and $s_{12} = [12] \langle 21 \rangle$, we finally find

$$i \frac{\langle 12 \rangle^8 s_{12} \epsilon(1234)}{N(5)} \quad (4.42)$$

which is the same as the left hand side of (4.36), therefore proving the relation.

We now review what we have done. We have explicitly proven equation (4.36), as a relation between $\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ supergravity, but since the objects involved are independent of \mathcal{N} , the relation is also true for SYM with \mathcal{N} supersymmetries and supergravity with $\mathcal{N} + 4$ supersymmetries. We can trace that backwards however, to equation (4.30), since all the subsequent steps were independent of \mathcal{N} (we only used group theory and the properties of the numerator coefficients β_{ijklm} , which are still those of $\mathcal{N} = 4$ SYM). In turn, (4.30) came about from the consistency condition of matching the IR behaviors of (4.25), (4.26). Since the left hand side of (4.25) is the known IR divergence of gravity amplitudes, still valid for $\mathcal{N} + 4$ supergravity, we can consider it as a check that the IR divergence of $A_{fg;lmn,\mathcal{N}}^{(1,1)}$ is still given by

$$\frac{r_{\Gamma}}{\epsilon^2} \sum_{i < j} (-s_{ij})^{-\epsilon} \sum_{abc \neq i, j} \epsilon_{lmn} [A_{ijabc}^{(0)}] \quad (4.43)$$

5 Conclusions

In this paper, we have shown that various linear relations between amplitudes of $\mathcal{N} = 8$ supergravity and the most-subleading-color amplitudes of $\mathcal{N} = 4$ SYM that were proven in refs. [16, 18] remain valid for the analogous amplitudes of $\mathcal{N} + 4$ supergravity and \mathcal{N} SYM theory, for any $\mathcal{N} \leq 4$. Specifically, the one- and two-loop four-point amplitudes of $\mathcal{N} + 4$ supergravity obey the relations

$$\mathcal{M}_{\mathcal{N}+4}^{(1)}(1, 2, 3, 4) = \frac{1}{2} \left(\frac{\kappa}{2} \right)^4 i s t A_{1234}^{(0)} A_{12;34,\mathcal{N}}^{(1,1)} \quad (5.1)$$

$$\mathcal{M}_{\mathcal{N}+4}^{(2)}(1, 2, 3, 4) = -\frac{1}{6} \left(\frac{\kappa}{2} \right)^6 i s t A_{1234}^{(0)} \left[u A_{1234,\mathcal{N}}^{(2,2)} + t A_{1342,\mathcal{N}}^{(2,2)} + s A_{1423,\mathcal{N}}^{(2,2)} \right] \quad (5.2)$$

which together can be rewritten as

$$\left(\sqrt{2} g^2 N \right)^L M_{SG,\mathcal{N}+4}^{(L)}(s, t) = \frac{1}{3} \left[\left(\left(\frac{\kappa}{2} \right)^2 u \right)^L M_{SYM,\mathcal{N}}^{(L,L)}(s, t) + \text{cyclic perms of } s, t, u \right] \quad (5.3)$$

valid for $L = 0, 1, 2$. The one-loop five-point amplitudes satisfy

$$\mathcal{M}_{\mathcal{N}+4}^{(1)}(1, 2, 3, 4, 5) = \frac{1}{20} \left(\frac{\kappa}{2}\right)^5 \sum_{S_5} i\beta_{12345} A_{12;345,\mathcal{N}}^{(1,1)} \quad (5.4)$$

We have also shown that the leading IR divergences of the subleading-color amplitudes $A^{(L,k)}$ for generic $SU(N)$ gauge theories are identical to those found for $\mathcal{N} = 4$ SYM theory [16, 17].

We note that the results in this paper are valid for $\mathcal{N} = 0$ YM, that is, pure glue theory, and therefore of possible interest for real world calculations. In particular, some of the results for IR divergences will generalize to QCD; this was already known to happen in some cases.

A central theme of this paper and our previous work on this subject is the matching of IR divergences of supergravity to those of the most-subleading-color amplitudes of SYM theory. In detail, the BCJ color-kinematic duality has provided an essential tool in proving some of these relations. It remains a challenge to exploit this point of view in cases when the BCJ numerators functions do not come outside of the integrals. Further generalizations to higher loops and/or n -points are difficult, but we can hope they are still possible.

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References

- [1] Z. Bern, J. J. M. Carrasco and H. Johansson, “*New Relations for Gauge-Theory Amplitudes*”, Phys. Rev. D78, 085011 (2008), [arxiv:0805.3993](#).
- [2] Z. Bern, J. J. M. Carrasco and H. Johansson, “*Perturbative Quantum Gravity as a Double Copy of Gauge Theory*”, Phys. Rev. Lett. 105, 061602 (2010), [arxiv:1004.0476](#).
- [3] Z. Bern, T. Dennen, Y.-t. Huang and M. Kiermaier, “*Gravity as the Square of Gauge Theory*”, Phys. Rev. D82, 065003 (2010), [arxiv:1004.0693](#).
- [4] J. J. M. Carrasco and H. Johansson, “*Generic multiloop methods and application to $N=4$ super-Yang-Mills*”, [arxiv:1103.3298](#).
- [5] J. J. M. Carrasco and H. Johansson, “*Five-Point Amplitudes in $N=4$ Super-Yang-Mills Theory and $N=8$ Supergravity*”, [arxiv:1106.4711](#).
- [6] Z. Bern, C. Boucher-Veronneau and H. Johansson, “ *$N \geq 4$ Supergravity Amplitudes from Gauge Theory at One Loop*”, [arxiv:1107.1935](#).
- [7] C. Boucher-Veronneau and L. Dixon, “ *$N \geq 4$ Supergravity Amplitudes from Gauge Theory at Two Loops*”, [arxiv:1110.1132](#).
- [8] S. Weinberg, “*Infrared photons and gravitons*”, Phys. Rev. 140, B516 (1965).
- [9] D. C. Dunbar and P. S. Norridge, “*Infinites within graviton scattering amplitudes*”, Class. Quant. Grav. 14, 351 (1997), [hep-th/9512084](#).
- [10] S. G. Naculich and H. J. Schnitzer, “*Eikonal methods applied to gravitational scattering amplitudes*”, JHEP 1105, 087 (2011), [arxiv:1101.1524](#).
- [11] C. D. White, “*Factorization Properties of Soft Graviton Amplitudes*”, JHEP 1105, 060 (2011), [arxiv:1103.2981](#).
- [12] R. Akhoury, R. Saotome and G. Sterman, “*Collinear and Soft Divergences in Perturbative Quantum Gravity*”, [arxiv:1109.0270](#).
- [13] S. G. Naculich, H. Nastase and H. J. Schnitzer, “*Two-loop graviton scattering relation and IR behavior in $N=8$ supergravity*”, Nucl. Phys. B805, 40 (2008), [arxiv:0805.2347](#).
- [14] A. Brandhuber, P. Heslop, A. Nasti, B. Spence and G. Travaglini, “*Four-point Amplitudes in $N=8$ Supergravity and Wilson Loops*”, [arxiv:0805.2763](#).
- [15] Z. Bern, J. S. Rozowsky and B. Yan, “*Two-loop four-gluon amplitudes in $N = 4$ super-Yang- Mills*”, Phys. Lett. B401, 273 (1997), [hep-ph/9702424](#).
- [16] S. G. Naculich, H. Nastase and H. J. Schnitzer, “*Subleading-color contributions to gluon-gluon scattering in $N=4$ SYM theory and relations to $N=8$ supergravity*”, JHEP 0811, 018 (2008), [arxiv:0809.0376](#).
- [17] S. G. Naculich and H. J. Schnitzer, “*IR divergences and Regge limits of subleading-color contributions to the four-gluon amplitude in $N=4$ SYM Theory*”, JHEP 0910, 048 (2009), [arxiv:0907.1895](#).
- [18] S. G. Naculich and H. J. Schnitzer, “*One-loop SYM-supergravity relation for five-point amplitudes*”, [arxiv:1108.6326](#).
- [19] Z. Bern and D. A. Kosower, “*Color decomposition of one loop amplitudes in gauge theories*”, Nucl.Phys. B362, 389 (1991).
- [20] V. Del Duca, A. Frizzo and F. Maltoni, “*Factorization of tree QCD amplitudes in the high-energy limit and in the collinear limit*”, Nucl. Phys. B568, 211 (2000), [hep-ph/9909464](#).

- [21] V. Del Duca, L. J. Dixon and F. Maltoni, “*New color decompositions for gauge amplitudes at tree and loop level*”, Nucl. Phys. B571, 51 (2000), [hep-ph/9910563](#).
- [22] Z. Bern, L. J. Dixon, D. C. Dunbar, M. Perelstein and J. S. Rozowsky, “*On the relationship between Yang-Mills theory and gravity and its implication for ultraviolet divergences*”, Nucl. Phys. B530, 401 (1998), [hep-th/9802162](#).
- [23] H. Nastase and H. J. Schnitzer, “*Twistor and Polytope Interpretations for Subleading Color One-Loop Amplitudes*”, [arxiv:1104.2752](#).
- [24] Z. Bern, L. J. Dixon, D. C. Dunbar and D. A. Kosower, “*One-Loop n -Point Gauge Theory Amplitudes, Unitarity and Collinear Limits*”, Nucl. Phys. B425, 217 (1994), [hep-ph/9403226](#).
- [25] S. Catani and M. H. Seymour, “*The Dipole Formalism for the Calculation of QCD Jet Cross Sections at Next-to-Leading Order*”, Phys. Lett. B378, 287 (1996), [hep-ph/9602277](#).
- [26] S. Catani, “*The singular behaviour of QCD amplitudes at two-loop order*”, Phys. Lett. B427, 161 (1998), [hep-ph/9802439](#).
- [27] S. G. Naculich, “*All-loop group-theory constraints for color-ordered $SU(N)$ gauge-theory amplitudes*”, [arxiv:1110.1859](#).
- [28] E. W. N. Glover, C. Oleari and M. E. Tejeda-Yeomans, “*Two-loop QCD corrections to gluon gluon scattering*”, Nucl. Phys. B605, 467 (2001), [hep-ph/0102201](#).
- [29] G. Sterman and M. E. Tejeda-Yeomans, “*Multi-loop amplitudes and resummation*”, Phys. Lett. B552, 48 (2003), [hep-ph/0210130](#).
- [30] S. M. Aybat, L. J. Dixon and G. F. Sterman, “*The two-loop soft anomalous dimension matrix and resummation at next-to-next-to leading pole*”, Phys. Rev. D74, 074004 (2006), [hep-ph/0607309](#).
- [31] Z. Bern, L. J. Dixon and D. A. Kosower, “*One loop corrections to five gluon amplitudes*”, Phys. Rev. Lett. 70, 2677 (1993), [hep-ph/9302280](#).
- [32] R. Kleiss and H. Kuijf, “*Multi-gluon cross-sections and five jet production at hadron colliders*”, Nucl. Phys. B312, 616 (1989).
- [33] H. Nastase and H. J. Schnitzer, “*On KLT and SYM-supergravity relations from 5-point 1-loop amplitudes*”, JHEP 1101, 048 (2011), [arxiv:1011.2487](#).